PROCLUS' DIVISION OF THE MATHEMATICAL PROPOSITION INTO PARTS: HOW AND WHY WAS IT FORMULATED?[1]

1. MOTIVATION

There are a number of ways in which Greek mathematics can be seen to be radically original. First, at the level of mathematical contents: many objects and results were first discovered by Greek mathematicians (e.g. the theory of conic sections). Second, Greek mathematics was original at the level of logical form: it is arguable that no form of mathematics was ever axiomatic independently of the influence of Greek mathematics. Finally, third, Greek mathematics was original at the level of form, of presentation: Greek mathematics is written in its own specific, original style. This style may vary from author to author, as well as within the works of a single author, but it is still always recognizable as *the* Greek mathematical style. This style is characterized (to mention a few outstanding features) by (i) the use of the lettered diagram, (ii) a specific technical terminology, and (iii) a system of short phrases ('formulae'). I believe this third aspect of the originality—the style—was responsible, indirectly, for the two other aspects of the originality. The style was a tool, with which Greek mathematicians were able to produce results of a given kind (the first aspect of the originality), and to produce them in a special, compelling way (the second aspect of the originality). This tool, I claim, emerged organically, and reflected the communication-situation in which Greek mathematics was conducted. For all this I have argued elsewhere.[2]

Other writers had been much less charitable towards this style. The greatest historian of Greek mathematics in the twentieth century, Wilbur Knorr, believed this style to be the result of the later imposition of philosophers. It was, according to him, a system invented by the derivative minds of Alexandrine scholastic intellectuals, stifling the thought of creative mathematicians.[3] So, according to Knorr's view, the style in which Greek mathematics is written is a result of an explicit codification from the outside: a style manual. This is very far from my view, of an organic development of the style.

Generally, the style of Greek mathematics attracted little attention (excluding passing, often negative remarks, such as Knorr's). The literature, both ancient and modern, has been more focused on the contents and the logical forms, not on the stylistic forms. However, there is one component in the Greek mathematical style which received some attention in the past. This is Proclus' scheme of the analysis of the mathematical proposition into six parts. (I will describe this scheme in the next section.)

Proclus asserts that this scheme holds for 'every problem and theorem' (*In Eucl.*

[1] I wish to thank Sir Geoffrey Lloyd, as well as an anonymous reader, for helpful comments.

[2] R. Netz, *The Shaping of Deduction in Greek Mathematics* (Cambridge, forthcoming).

[3] W. Knorr, 'On the early history of axiomatics: the interaction of mathematics and philosophy in Greek antiquity', in J. Hintikka *et al.* (edd.), *Theory Change, Ancient Axiomatics, and Galileo's Methodology* (Dordrecht, 1981) pp. 145–86, esp. pp. 172ff. The main interest of Knorr there is large-scale style (the organization of treatises), not small-scale style (the organization of individual propositions).

203.1–2), and this claim has never, to my knowledge, been challenged in the past.[4] This is the most visible component of the Greek mathematical style, at least in the sense that no other component has been commented upon so often.

It therefore acquires a special importance from the perspective of our present interests. Since it is the most visible component, we can make the following *a fortiori* claim. If this component is not the result of some attempt at explicit codification—of something like a style manual—then probably no other component is. If the Greek mathematical style is a result of an explicit codification, then the place to find such explicit codifications would be this scheme. On the other hand, if this scheme is not the result of explicit codification, then probably nothing in the Greek mathematics style is. So the question of the origins of the Greek mathematical style is dependent upon the question of the origins of the Proclean division of the proposition.

It is very difficult to answer directly the historical question, whether the scheme offered by Proclus reflects some ancient explicit codification. But there are two relevant probable arguments.

1. The more we find that the actual practice answers to the scheme, the more probable it becomes to ascribe the practice to some explicit codification. On the other hand, if we find that the actual practice is loose, and does not follow simple rules, then it becomes more probable to think of the practice as evolving spontaneously, without explicit rules. So this is the argument based upon the *practice*.
2. The more we ascribe the *terms* used in the scheme to very ancient times, the more probable it becomes that the system was in fact introduced early on. It then becomes more probable that the scheme might have been introduced prescriptively, to regulate a practice, and not descriptively, to comment upon an already established practice. On the other hand, the more we ascribe the terms to late periods, closer in time to Proclus himself, the more probable it becomes that the system is a late, descriptive invention. So this is the argument based upon the *terminology*.

I shall first describe Proclus' scheme, in section 2. In section 3 I shall analyse the practice. In section 4 I shall analyse the terminology. Section 5 is a brief summary.

Before we get started, another motivation for the discussion ought to be mentioned. If we think that Proclus repeats an old system, designed prescriptively, this entails one picture of Proclus' enterprise: dependent on the past, hardly creative at all. On the other hand, if we suppose that he uses a newer system, designed descriptively, then we may begin to think of him as a more original writer (or at least as a representative of a more original tradition), and we may think of the scheme as an ingenious analytic tool, worthy of our applause. So part of the motivation is the question: 'did Greek mathematicians merely follow an externally imposed codification?'; another part is the question 'was Proclus (or his tradition) a mere compiler of earlier material?' One answer, and both the mathematicians and their commentator become uncreative followers of tradition; another answer, and both are found to be creative, original writers. But let us first look at the evidence.

[4] It is endorsed, for example, in I. Mueller, 'Greek mathematics and Greek logic', in J. Corcoran (ed.), *Ancient Logic and its Modern Interpretation* (Dordrecht, 1974), p. 67, n. 4. D. Fowler notes in *The Mathematics of Plato's Academy* (Oxford, 1987), p. 367, n. 54, that it may not have been always strictly followed.

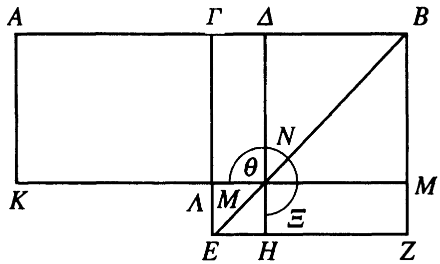

FIGURE 1

2. THE PROCLEAN SCHEME

The division of the proposition into parts is the subject of *In Eucl.* 203.1–207.25. The context is Proclus' commentary to the first proposition of the *Elements*. This is a natural point in which to give general analytic tools, and the main tool Proclus offers is that of the division into parts. 203.1–207.25 discuss the scheme in general; following that, the scheme is used in the detailed commentary to the first proposition;[5] and then it is taken up now and again, in connection with other propositions (e.g. at 296.15–18).

It so happens that the first proposition is not an ideal case for describing the scheme, for reasons to be explained later. I shall therefore take another simple proposition, *Elements* 2.5, as my case-study. I give a translation, with headings following the Proclean scheme. Then I shall briefly explain the scheme, following Proclus as closely as possible.

2.1 Euclid's Elements *2.5*

See Figure 1.

[*protasis* (enunciation)]
If a straight line is cut into equal and unequal <segments>, the rectangle contained by the unequal segments of the whole, with the square on the <line> between the cuts, is equal to the square on the half.

[*ekthesis* (setting out)]
For let some line, <namely> the <line> *AB*, be cut into equal <segments> at the <point> *Γ*, and into unequal <segments> at the <point> *Δ*;

[*diorismos* (definition of goal)]
I say that the rectangle contained by the <lines> *AΔ*, *ΔB*, with the square on the <line> *ΓΔ*, is equal to the square on the <line> *ΓB*.

[*kataskeue* (construction)]
For, on the <line> *ΓB*, let a square be set up, <namely> the <square> *ΓEZB*, and let the <line> *BE* be joined, and, through the <point> *Δ*, let the <line> *ΔH* be drawn parallel to either of the <lines> *ΓE*, *BZ*, and, through the <point> *Θ*, again let the <line> *KM* be drawn parallel to either of the <lines> *AB*, *EZ*, and again, through the <point> *A*, let the <line> *AK* be drawn parallel to either of the <lines> *ΓΛ*, *BM*.

[5] For example, 208.4–15 describe the *protasis* of the first proposition, and explain how it fits the general description of *protasis*.

[*apodeixis* (proof)]
And since the complement $\Gamma\Theta$ is equal to the complement ΘZ; let the <square> ΔM be added <as> common; therefore the whole ΓM is equal to the whole ΔZ. But the <area> ΓM is equal to the <area> $A\Lambda$, since the <line> $A\Gamma$, too, is equal to the <line> ΓB; therefore the <area> $A\Lambda$, too, is equal to the <area> ΔZ. Let the <area> $\Gamma\Theta$ be added <as> common; therefore the whole $A\Theta$ is equal to the gnomon $MN\Xi$. But the <area> $A\Theta$ is the <rectangle contained> by the <lines> $A\Delta$, ΔB; for the <line> $\Delta\Theta$ is equal to the <line> ΔB; therefore the gnomon $MN\Xi$, too, is equal to the <rectangle contained> by the <lines> $A\Delta$, ΔB. Let the <area> ΛH be added <as> common (which is equal to the <square> on the <line> $\Gamma\Delta$); therefore the gnomon $MN\Xi$ and the <area> ΛH are equal to the rectangle contained by the <lines> $A\Delta$, ΔB and the square on the <line> $\Gamma\Delta$; but the gnomon $MN\Xi$ and the <area> ΛH, <as a> whole, is the square ΓEZB, which is <the square> on the <line> ΓB; therefore the rectangle contained by the <lines> $A\Delta$, ΔB, with the square on the <line> $\Gamma\Delta$, is equal to the square on the <line> ΓB.

[*sumperasma* (conclusion)]
Therefore if a straight line is cut into equal and unequal <segments>, the rectangle contained by the unequal segments of the whole, with the square on the <line> between the cuts, is equal to the square on the half; which it was required to prove.

2.2 *An exposition of the Proclean scheme*

The scheme consists of six parts: *protasis*, *ekthesis*, *diorismos*, *kataskeue*, *apodeixis*, and *sumperasma*. I shall discuss the literal meaning of each term later on in the article.

As Proclus says, the *protasis* has a binary structure (which Proclus defines at 203.6—to paraphrase—as 'given a certain thing, what is that we look for'). In the ideal case, it states a condition, and then states that some other result follows under that condition—but there are many variations, as hinted at even by Proclus.[6] In the case of 2.5, the condition is the line divided into equal and unequal segments. The result is a specific geometrical equality.

Proclus, for some reason, does not dwell upon what we may see as the main characteristic of the *protasis*, namely its being general. Whatever the reason for Proclus' omission, it is this generality which distinguishes the *protasis* from the immediately following two parts. The *protasis* has, as explained, two parts: a condition, and a result, both general. The next part, the *ekthesis*, sets a particular condition, and the third part, the *diorismos*, sets a particular result. In 2.5, the *ekthesis* divides a particular line into particular equal and unequal segments. The *diorismos* states a particular geometrical relation then holding between objects based on those mentioned in the *ekthesis*.

The purpose of this structure is a difficult problem. Why general and particular, mirroring each other? And why a binary structure of condition and result? This problem is only indirectly related to the origins of the scheme, and this paper will say nothing on it.[7]

Proceeding further in the scheme, the two following parts are much more obvious. The *diorismos* stated the desired result for the particular case, but stating is not proving. What is required now is a proof. However, as explained by Proclus, sometimes the given (i.e. the construction implicit in the *ekthesis*) is not sufficient as a basis for the proof (203.10–12). In the case of 2.5, the *ekthesis* gives the lines, but not yet the squares and rectangles built on the lines. Those geometrical areas are merely potential,

6 'for the perfect *protasis* consists of both' (203.7–8).
7 I have dealt with this in ch. 6 of Netz (n. 2).

as far as the *ekthesis* goes. To make the proof at all possible, then, the objects of proof must be constructed, and this is done in the *kataskeue*. So this is the fourth part.

Then finally one gets the proof, the *apodeixis*. This is a sustained argument about the particular objects of the diagram, which ends when the result stated in the *diorismos* is obtained.

To this is appended the *sumperasma*. This returns to the *protasis*, asserting that it has been proved. To be more precise: the *sumperasma* is nothing but a repetition of the *protasis*, with the word 'therefore' added, and the Greek words meaning 'Q.E.D.' appended. It is thus the most formal, rigid part of the proposition.

These are the six parts, perfectly instantiated in 2.5, in their right order, and doing what Proclus expects them to do. How often is the scheme so simply instantiated?

3. THE PRACTICE

Before discussing the actual variability in the practice, it is useful to consider in general in what ways the practice *may* be variable in principle. The Proclean scheme consists of six consecutive parts, and may be succinctly given as the rule '123456'. A variation from such a rule may take the principle forms: either missing parts ('12456', for instance), or a permutation of the parts ('142356', for instance), or of course some combination of the two. Finally there is the possibility of some added parts, not governed by the scheme ('1237456', for instance), but this is less clearly a *violation* of the rule.

In fact, permutations in the strict sense do not occur, and this is easy to explain. The sequence, given the elements, is the only one possible, even logically. Take, for instance, the last pair of parts, '56', 'proof–conclusion'. It is a logical necessity that the conclusion will follow the proof, for the reason that had it been otherwise, had the conclusion come first, we simply would not *call* it conclusion. Part of what makes conclusions what they are is their relative position to proofs, their position following proofs. Otherwise they are mere assertions, 'enunciations'—and this is the first part of the proposition. Or take the pair before that, '45', 'construction–proof'. Since the proof must refer to the construction, it is mathematically meaningless to have all of the construction coming after all the proof. And similar cases can be argued for the rest of the scheme. We begin to see by such considerations how the scheme may be the result of spontaneous development, not of explicit codification.

However, as I have already suggested concerning the relation of construction and proof, while strict permutation is hardly an option, something similar may happen, and this is the intermingling of parts. Construction and proof may interpenetrate, and need not be separate, distinct parts. I shall clarify this with examples below.

So the possible variability we see may be either that of missing parts, or of interpenetration of parts. I shall say something on both (and on the possibility of added parts), but before that it is necessary to see the extent Proclus himself allowed the scheme to be variable.

3.1 *The variability according to Proclus*

Proclus' discussion of the allowed variability begins with following general statement:[8]

[8] Here and elsewhere in this article I adapt Morrow's translation with a few changes, mainly in changing the names of the parts where mine are different from his.

The [parts] which are always present are enunciation, proof and conclusion . . . [T]he other parts are . . . often left out when they serve no need. For both setting-out and definition of goal are omitted in the problem 'To construct an isosceles triangle having each of its base angles double the other angle'. And in most theorems there is no construction, because the setting-out is sufficient . . . (203.17–204.4)

In other words, Proclus does not consider variability except that of missing parts. First, the construction may be missing when 'the setting-out is sufficient', which is a very wide clause. So the construction is effectively optional.[9] But what about the other two parts which may be missing according to Proclus: setting-out and definition of goal? Proclus gives an example of a case where they are missing: a problem, Euclid's *Elements* 4.10. Now the example may serve simply as a textual reference, showing that the parts may be missing in a real Euclidean case, or it may be more significant, in showing us the *kind* of cases where those parts may be missing. It is clear that the last is what Proclus intends, for he immediately goes into a long discussion, starting at 204.5–6: 'When, then, do we say the setting-out is missing?'[10] Obviously Proclus does not mean to ask a *diagnostic* question—when are we *allowed* to say that the setting-out is missing? The diagnostic question is answered simply by looking at the proposition and seeing whether it has a setting-out or not. Proclus question can only be '*under what conditions* is the setting-out missing?' The point of the first-person 'we say' is probably, then, to assert that the following piece of analysis is Proclus' own contribution. We begin to see in what ways Proclus aims at originality.

The analysis which follows (204.6–205.12) is a bit vague, but on the whole Proclus' intention is clear and it fits well the case he quotes.[11] Briefly, the main point is that (204.9–10): 'Sometimes [the enunciation] states only what is sought, that is, what must be known or constructed'. That is, generally, enunciations are bipartite ('if . . . then [it is always the case that] . . .' or 'given . . . then it is required to construct . . .'). Sometimes, however, they have only one part ('[it is always the case that] . . .' or 'it is required to construct . . .'). Then, Proclus says, there is no need of a setting-out, and as a result there is no need of a definition of goal.

Whatever we think of this analysis, one thing is clear: Proclus' view is that, through this analysis, he has identified the conditions under which the setting-out and definition of goal may be omitted. So we can sum up the variability of the scheme according to Proclus: constructions are effectively optional, but otherwise everything must be in place except that, when the enunciation has only one part, setting-out and definition of goal are dropped. Let us now look at the practice.

9 It is optional in a deep sense. It is very often possible to state the construction within the setting-out itself. For instance, nothing stopped Euclid, in *Elements* 2.5, from constructing the relevant areas on the line-segments within the setting-out. Then the definition of goal would have referred not to *virtual* objects, 'the rectangle contained by the lines . . .', but to actual rectangles and squares, and no construction would have been required. Very often, then, this is a purely stylistic decision, how to distribute the burden of construction between the setting-out and the construction (in other cases, of course, the added construction really is very clearly auxiliary to the construction demanded to set out the proposition itself: Pythagoras' theorem, *Elements* 1.47, is an obvious case).

10 Proclus sees here the definition of goal as dependent upon the setting-out, and thus he concentrates on whether the setting-out is present or not.

11 As well as others in the *Elements*. Proclus refers to Books 7–10 in general, and to 10.28 in particular, as furnishing further examples.

3.2 *Missing parts*

Of course, constructions are often missing, as Proclus allows (e.g. the first theorem in Euclid's *Elements*, 1.4, does not have a construction). But while constructions may be missing in theorems (whose goal is to show that a statement is true), they may not be missing in problems (whose goal is to bring about some mathematical object): just as proofs are essential to theorems, so constructions are essential to problems. Proclus' latitude is, in a way, too wide. It is not true to say that constructions are completely optional. And we begin to see that Proclus has in mind especially theorems.

What about his other concession, that setting-out and definition of goal may be missing, but only when the enunciation has just one part (instead of two)? Here even Euclid's practice (not to mention other mathematicians) is freer. Euclid's Book 13, for instance, has as its central object the construction of the five regular solids (constructed at propositions 13–17). In these five propositions, the enunciation has always the two parts which Proclus sets out, that which is given and that which is required to construct. The given is the radius of the sphere in which the solids are to be inscribed, and the task is to construct the inscribed solids. Yet Euclid does not give setting-out and definition of goal of the form 'Let the radius AB be given. So it is required to . . .' Instead, he proceeds directly to the construction of the solids.

Theorems are more conservative in this respect, but definitions of goal may still disappear for no obvious reasons. In Apollonius' *Conics* 4.34, for instance, the enunciation has the form 'If an ellipse touches an ellipse . . . at two points . . . then the line joining the points . . . falls through the centre'. That is, this is a typical bipartite enunciation, with separate conditions and results. The text then proceeds with what looks like a setting-out, with the objects constructed. Then however, instead of saying something like 'I say, that *XY* falls through the centre', the text simply asserts, as if this was part of the proof, 'Therefore *XY* falls through the centre'. And there follows a brief argument showing that this is true. In other words, the construction was not a proper setting-out, but was already the construction preceding the proof. This proposition then, for no clear reason, has no definition of goal, and its constructive phase is better seen as construction, not as setting-out.

This last example reminds us again of the crucial feature of the system: it is a system, a structure. When one object is missing or changes its nature, the entire system is affected. 'Setting-out' and 'construction' are defined not just by what they are, but also by their relation to other parts, such as the definition of goal. The terms are context-sensitive: when identifying a stretch of text as the Proclean part it is, we say something not only about this stretch, but about its context as well. Proof is least context-sensitive, but all the remaining parts, to different degrees, are context-sensitive. The result is that any variation in the practice changes the entire face of the proposition.

Take for instance Archimedes' *Sphere and Cylinder* 1.23. In this proposition only a single part is clearly missing, but this, however, is the enunciation: this proposition simply has no enunciation in general terms. Instead, it starts directly with a particular geometrical object. The first event of the proposition is that this object is geometrically produced. Since we have no enunciation, we cannot judge whether this production is formally a setting-out, or already a construction: the two are distinguished primarily by reference to an enunciation, which is missing here. Following this setting-out or construction, Archimedes makes an assertion (call it The Assertion), which he then sets out to prove (call this The Proof of the Assertion), with the aid of another auxiliary construction (call this The Construction for the Assertion). So it is possible to identify The Assertion as a definition of goal, The Construction for the Assertion as a construction, and The Proof of the Assertion as a proof. But The Assertion looks just like any assertion in the course of a proof (just as we saw with Apollonius' *Conics* 4.34), i.e. it is not preceded by a 'I say, that' or any equivalent expression. With the removal of the cornerstone of the Proclean scheme—the enunciation—the entire edifice collapses, the scheme becomes almost irrelevant. Of course there are constructions on the one hand, and arguments on the other hand. This, after all, is the stuff geometry is made of. But no further distinctions are at all useful.

Archimedes' *Sphere and Cylinder* 1.23 is not a unique case. This is a form Archimedes deploys a number of times in the same book (in the later propositions 28, 36, 39, 41). So this alerts us to the possibility that there might be other forms, rivals to the Proclean scheme. We shall return to this possibility below.

So we saw cases where setting-out and definition of goal may be missing, although Proclus would not allow them to be missing, and we see that even the enunciation may be missing. We also saw that Proclus' statements concerning constructions are not precise, but in this case he errs in the direction of allowing more variability than is in fact the case. Remaining parts which we have not mentioned so far as missing are the last two, the proof and the conclusion. Indeed, proofs are necessary for propositions, but perhaps this represents a convention of how we count propositions. Statements which are not followed by proofs occur in Greek mathematics: these are the so-called corollaries, which often (though not always) simply assert that such and such is obvious on the basis of the preceding argument. If we count such bits of text as 'propositions', then these are propositions in which everything is missing except the enunciation (they are thus the direct opposite of Archimedes' *Sphere and Cylinder* 1.23). The only reason not to count them as propositions is their title, *porisma*, which sets them apart from 'normal' propositions. However, such titles often have scant textual authority, and may thus represent the fussiness of editors, worried about standardizing the texts, and I do not mean necessarily ancient editors: in Archimedes, almost all such titles were introduced by the modern editors. Thus the original text may be seen as made not of 'propositions' and 'corollaries', but of different kinds of propositions, some with proofs and some without.

Leaving this possibility aside, the remaining part is the conclusion. Here the Proclean scheme is at its weakest: even Euclid often leaves out the conclusion. To begin with, he usually leaves it out in problems, and does not return to the enunciation, stating in general terms 'therefore an object of such-and-such a description has been constructed'. This is because a problem does not seek out to construct an object in general—how could it? Whatever is constructed is particular, and therefore what the problem seeks to do is to construct some particular object answering the description. When this has been done, there is nothing more to say, except the phrase 'which it was required to do'. In some manuscripts, however, there are some cases where the conclusion of a problem does revert to the general enunciation. Most notably, this happens in the first proposition of the *Elements*—in the course of whose commentary the scheme is offered! But this in fact is highly revealing. *The only textual authority for a conclusion in this proposition is Proclus himself* (and therefore even Heiberg finally prefers to doubt it). Proclus tampered with his own manuscript, to make it agree with the scheme he was offering.

Elsewhere in the *Elements*, the conclusion is often missing even from theorems (this happens in most propositions in the arithmetical Books 7–9, for instance). It hardly

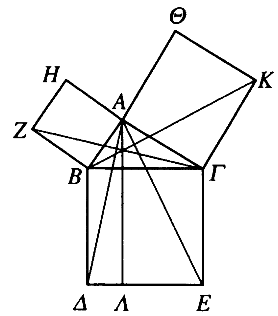

FIGURE 2

ever appears outside Euclid's *Elements*; after all it is in a sense redundant. As long as one is willing to believe that the proof for the particular case yields a proof for the general case, there is no need for a conclusion—the proof already reaches the particular conclusion. And if one is not willing to make this leap, the conclusion is of no help, since it does not *do* anything to bridge the gap between the particular and the general: it simply *asserts* the general result. But, whatever our philosophical views, the important fact is not only that the conclusion is most often omitted, but that Proclus insists that it cannot be omitted; in fact he goes against his own text in this insistence. So what is his project? Certainly he concentrates upon Euclid's *Elements*. When he points out the possible omission of setting-out and definition of goal, he does refer to books other than the first. But in this insistence about the necessity of conclusion, Proclus probably has only the theorems of Book 1 in mind. A picture seems to emerge: Proclus has in mind, predominantly, theorems, and predominantly Book 1 of the *Elements*. Secondarily, he considers problems, and the remaining of the *Elements*. He does not consider other mathematical treatises at all. Is this picture valid?

3.3 How parts intermingle

I suggest that Proclus has in mind predominantly Book 1 of the *Elements*, but even here there are exceptions. I take proposition 47 (Figure 2):

[Enunciation] In right-angled triangles, the square on the hypotenuse is equal to the squares on the sides containing the right angle. [Setting-out] Let there be a right-angled triangle, <namely> $AB\Gamma$, etc. . . . [Definition of goal] I say that the square on $B\Gamma$ is equal to the squares on BA, $A\Gamma$. . . . [Construction] For let a square, <namely> $B\Delta E\Gamma$, be constructed on $B\Gamma$, etc. . . . [Proof] And since each of the angles <contained> by $BA\Gamma$, BAH are right, etc. . . .

So far, we have gone through the five first parts, in their right order, and in their right format: they are compartmentalized, sealed off from each other. What we expect now is that the proof will go on, terminate, and then we shall have a conclusion. However, near the end of the proof we have the following (112.18–20): 'So, joining the <lines> AE, BK, it shall be similarly proved that the parallelogram $\Gamma\Lambda$, too, is equal to the square $\Theta\Gamma$'.

What is this? Construction perhaps? After all it does join two lines which were not joined anywhere else in this proposition. Or is it part of the proof? It appears within the proof, and it does make an assertion. The truth is that the question, 'What part is it?' is only meaningful starting from Proclus' schema. Starting from the text itself, it does not lend itself naturally to such divisions, because parts do not come in such neat bits.

The most typical way in which such things happen is when the construction is spread through the proposition, not just at one particular stage prior to the proof. Proclus implies it should come as one neat stage. What is more important for our purposes, it ought to have come in one neat stage, had the structure of the proposition been laid down as 'Propositions must have the following structure'. Had there been a style manual, explicitly codifying the Proclean scheme, it could only be set out in terms of one part coming after the next. For without this, all we have is that the proposition ought to include elements such as construction and proof, but not necessarily in this order and not necessarily separated from each other. But this is just saying nothing: of course you must have construction and proof in geometry.

We begin to see the following distinction. Concepts such as 'construction' and 'proof' can mean either a conceptual unit, or a stylistic unit. As a conceptual unit, saying that 'a proposition P has a construction' means simply that objects are constructed within this proposition, and this is a description of the logical nature of the proposition. As a stylistic unit, saying that 'a proposition P has a construction' must mean that there is a specific stage in the proposition where objects are constructed. There must be a specific chunk of text, a part of the proposition, which is 'The Construction'. And if construction and proof regularly intermingle, there is no longer a *stylistic* unit such as the construction or, indeed, the proof.

So perhaps Proclus himself in his scheme uses the parts of the propositions as merely conceptual terms? Perhaps the scheme was never stylistic, but logical? I shall return to discuss Proclus' intentions in the summary. What I wish to argue right now is that, even if Proclus intends the scheme as a stylistic scheme, this is not an adequate description of the structures of propositions.

For such an intermingling as *Elements* 1.47 often happens. I had to look towards the end of the book, but this is because Book 1 generally (as is expected) has very simple constructions, which are over within a sentence or two and thus, naturally, are not spread over the whole proposition. But elsewhere this phenomenon—the spread construction—is common. Take, for instance, *Elements* 2.10: lines 146.27–148.4 are construction, lines 148.4–10 are proof, 148.10–11 are construction again, 148.11 to the end are proof again. The construction at the middle of the proof, 148.10–11, is essentially linked to the proof, because its possibility has to be proved—and this is a typical complication. And then inside problems, the relations may be even more complex. In a problem, where the task is to construct a geometrical object, the 'proof' is a reflection upon the construction. The relation between construction and proof is much more intimate in a problem than in a theorem. So one has passages such as in *Elements* 2.14 (Figure 3):

> [Construction] For let the right-angled parallelogram $B\Delta$ be set up equal to the rectilinear <figure> A; [Proof] now first, if BE is equal to $E\Delta$, that which is to be constructed will have come to be. For a square has been set up . . . and if not, then . . . [Construction] let <BE> be produced to Z, etc. . . . [Proof] Now since a line, <namely> BZ, has been cut, etc. . . .

The mathematician makes a drawing, pauses to consider it in a brief argument, moves on to another act of drawing—the two elements of the mathematical practice, drawing and arguing, cannot be set apart into neat compartments.

We see that construction and proof intermingle, and these two parts are set apart from the others, in that they both (especially the proof) can be very long. A construc-

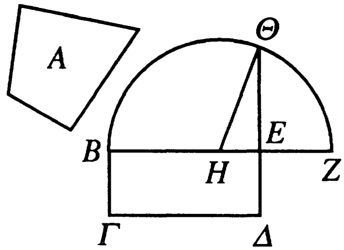

FIGURE 3

tion is one object being drawn after another, a proof is one statement being made after another, and both may be developed in principle indefinitely: they are like a development of a musical theme. The enunciation, setting-out, definition of goal and conclusion are much more confined. The enunciation is the introduction of the main theme; the setting-out and definition of goal are the introduction of the secondary theme; the conclusion restates the main theme. There is no development in these parts, therefore they are briefer, and cannot be spread in the same way. Even more remarkable, then, that they are in fact quite often broken down into smaller bits, and are thus spread through the proposition. *Elements* 3.8, for instance, has a rather complex enunciation: it makes simultaneously a number of assertions and, instead of translating immediately all those assertions into particular terms, in a single definition of goal, Euclid divides the definition of goal into two, more manageable parts. He first has one 'I say that . . .' clause, in 184.13–186.2, and then he has another, in 188.5–7. The resulting structure of the entire proposition is:

Enunciation—Setting-out—Definition of Goal—Construction—Proof—Definition of Goal—Construction—Proof—Conclusion

Again, we see that the parts, as conceptual units, are still useful—conceptually. But are they stylistic units? It is hard to think of them in this way, because nothing prevented Euclid from running the two definitions of goal together. The definition of goal is divided into two, for the sake of mere convenience, and so it is very unlikely that there is any importance attached to it as a stylistic unit. There is no attempt to have a chunk of the proposition which is dedicated to definition of goal.

I have concentrated so far on examples from the early books of the *Elements*, books Proclus should have known well, and of course many more examples could have been brought from elsewhere.[12] But let us concentrate on what Proclus should have known. Already in the ninth proposition of Book 1, there is a (very typical) structure which ought to have given Proclus a pause. This is a problem (Figure 4):

[Enunciation] To bisect the given rectilinear angle. [Setting-out] Let the given rectilinear angle be the <angle contained> by *ΒΑΓ*. [Definition of goal 1] So it was required to bisect it. [Construction] Let a chance point, *Δ*, be taken etc. . . . [Definition of goal 2] I say that the angle <contained> by *ΒΑΓ* is bisected by the line *ΑΖ*.

And then follow proof and conclusion.

[12] For example, the division of the definition of goal into several parts, spread through the proposition, is very common in Autolycus: six out of the thirteen propositions of the first book of *The Risings and Settings*.

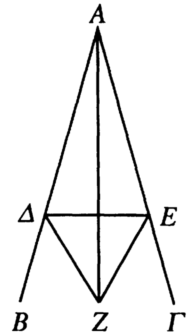

FIGURE 4

What we see here is that all problems fall outside Proclus' scheme taken strictly. A problem has not one goal, but two. One goal is purely constructive—to construct a given object. The second goal is argumentative—to show that the object fulfils the criteria. Often, as in here, the second goal is set out explicitly. Sometimes it is left implicit (as—conveniently for Proclus—happens in the first proposition of the *Elements*). It is not precisely that the definition of goal is spread through the proposition in such cases. More precisely, the Proclean scheme breaks down because it does not fit the actual, complex structure of problems.

We already saw the inadequacy of the Proclean scheme as regards problems in the preceding section: problems, naturally, do not have conclusions. Again, we see that Proclus has in mind especially one particular format of Greek mathematics, the theorem.

But there is a much wider field out there, which he ignores completely. I do not refer to structures which are obviously different in nature from the Euclidean project—e.g. discursive astronomical or mechanical texts, or even Diophantus' arithmetic. There is a clear sense in which Proclus need not have them in mind when he speaks about 'every problem and theorem'. But is it as legitimate to ignore structures which are essential to the ancient geometrical tradition, such as that of analysis and synthesis? The analysis and synthesis mode of proof has a stable format. One starts from assuming the goal as obtained. From this assumption, some known results are obtained. Then, on the basis of these known results, the goal is constructed or proved.[13] So here is a structure that is radically different from the Proclean scheme. The same conceptual elements are there, of course—there is construction and proof, certainly. But the format has nothing to do with Proclus. And this structure is, for instance, the only one used in the problems of the second book of Archimedes' *Sphere and Cylinder*. It does not occur in the main text of the Elements as printed by Heiberg, but it does appear in the manuscripts of Euclid's *Elements*, in Book 13, where there are five propositions written in the analysis/synthesis style.[14] So when he ignores the analysis/synthesis

[13] I cannot do justice in this brief compass to the *logical* detail of the method; all I need to stress is that the method has a standard *stylistic* format (which however, again, is not absolutely rigid). On the logical side, see especially the discussion in J. Hintikka and U. Remes, *The Method of Analysis* (Dordrecht, 1974).

[14] Printed as Appendix 1.8 in Heiberg's edition of *Elements* 13 (364.17–376.22).

technique, Proclus ignores what he probably considered to be Euclid. Proclus does refer, in a general way, to the analysis/synthesis technique (e.g. *In Eucl.* 67.7–8), but he does not mention it in the context of his scheme.

Or consider the following type of structure, again ignored by Proclus.

While Euclid does not compartmentalize tightly the various parts of the proposition, he is very good at another kind of compartmentalization: that between propositions. Every proposition in Euclid's elements is a fresh start. This is not the case generally in Greek mathematics, and indeed in many cases it is difficult to see where one proposition ends and another begins. The numbering of propositions often varies between manuscripts, and thus we cannot use it as a safe guide to the original division into propositions; perhaps there was no original division into propositions. At any rate, when propositions are not tightly sealed between them, this has implications for their internal structure, as well. Apollonius' *Conics* 1.52, for instance, solves the problem of constructing a parabola, with a certain given angle being right. Proposition 53 then starts off with 'The same things assumed, let the given angle not be right'. Whereupon Apollonius plunges immediately into the construction. The sentence above sums everything: enunciation, setting-out, and definition of goal. Of course this need not be seen as a separate proposition at all. It may be seen as an annex of proposition 52.[15] But this again only serves to stress that the structures of propositions may be much more complex than Proclus allows.

To sum up, then: Greek mathematical propositions may be written in a variety of formats, the chief of which are theorems and problems. Most theorems in the first book of the *Elements* follow closely the Proclean scheme. Problems, and theorems outside the first book of the *Elements* (and certainly outside the *Elements*) are often radically different from the Proclean scheme. The conclusion is almost always missing. Setting-out and definition of goal may be missing where Proclus considers them essential. In some cases, indeed, even the enunciation is missing. Construction and proof intermingle freely, and thus cannot be considered as stylistic units, only as conceptual units. The definition of goal is often broken into parts, so again it is impossible to see it as a stylistic unit. And in some cases, most frequently in problems, the structures are simply different to those laid down by Proclus. The practice of the theorems of the first book of the *Elements* fits Proclus' scheme very well. Otherwise, the practice of Greek mathematics hardly fits it at all.

4. TERMINOLOGY[16]

So the practice does not fit the scheme. But this does not rule out the possibility that the scheme was introduced in order to regulate the practice. Perhaps it was introduced as a style manual, only it was a failed style manual, faced with rebellious authors. Even in this case, it will still remain true that the style we actually possess of Greek mathematics, then, is not the result of explicit codification. But the question remains, whether such an explicit codification was ever attempted. This possibility

[15] The heading '53' simply does not occur in our main manuscript (it does occur in the margins of another). It was added by Heiberg, who followed the ancient commentator Eutocius—who of course had inserted numbers for the sake of his commentary.

[16] In what follows, I speak of the way words are used 'in our literature'. Greek mathematics is poorly represented by the current CD-ROMs available for Greek literature and, even with computers and indices, categorical statements are never safe. I make categorical statements. I know I may be wrong in each of my statements, and that probably I am wrong in some. I hope my mistakes will be pointed out—and I believe my main argument will still stand.

cannot be ruled out completely, but its probability can be assessed by looking at the origins of the terminology. I have introduced the distinction between the parts as conceptual units, and the parts as stylistic units. The more we find that the Greek terms for the parts are used as conceptual, and not as stylistic terms, the less likely it is that the scheme was originally introduced as a style manual. Even more damning will be to discover that the terms were not used in early periods at all, or were even used with a completely different meaning. In this case, it is very unlikely that the terms were introduced as an early style manual, simply because they were not introduced at an early period at all.

So I shall discuss each of the six terms in turn. But another, negative piece of evidence must first be mentioned. Whatever the individual histories of the individual terms, the system *as a system* is never attested before Proclus himself. It is not only that none of our sources refers to the entire scheme. Worse than this: none of our sources before Proclus refer to any *combination* of terms. We never have a statement such as, say, 'The setting-out repeats the enunciation'. In short, there is no evidence, prior to Proclus, for the terms seen from a structural point of view. I have explained above the structural nature of the scheme. A setting-out is only a setting-out when put in a certain relation, with other parts, such as the enunciation. If the terms are not seen as relational elements of a system, they cannot have the meanings Proclus attaches to them. Of course, the *e silentio* is not a strong argument in Greek mathematics. But it does mean that Aristotle and Pappus, the two extremes of the chronological framework, do not refer to the system, although both would have had plenty of occasion to make such references. On the other hand, Eutocius, later than Proclus, seems to use the Proclean scheme.[17] His words make his predecessors' silence more meaningful. However, let us look at the positive evidence: what are the individual terms taken to mean, and when?

4.1 Protasis *(enunciation)*

The Greek word *protasis* is the abstract noun of the verb *proteino*, 'to stretch forward, to put forward'. It is a loosely defined expression for action in space—the stuff of metaphor. Thus the verb and its cognates are highly polysemic.[18] We shall see this polysemy repeated with some other terms, and the same story can often be told for Greek technical terminology.

The word can easily mean either the conceptual unit or the stylistic unit. As the conceptual unit, it is 'that which is put forward' in the sense of 'proposal, claim'. It is that which the mathematician claims to be true (or possible)—the 'enunciation'. As the stylistic unit, it is 'that which is put forward' in the sense of 'that which is put in the beginning'. The ancients did not generally share our lexicographic anxieties, and the two senses were probably not separated out for them. Autolycus, for instance, may say, in the course of a proof (I offer, as I shall immediately explain, one of two possible translations):[19] 'And the remaining <parts> of the <claims made> by the *protasis* will be proved similarly'. The crucial words—'the <claims made> by the *protasis*'—read in the original τὰ διὰ τῆς προτάσεως, perhaps meaning 'the assertions made in the

[17] For instance, in the commentary to *Planes in Equilibrium*, 2.1, Eutocius clearly refers to 'enunciation' and to 'construction' as two elements of the *text*, i.e. as stylistic units.

[18] Some meanings on offer in LSJ: (verb) 'expose to danger', 'stipulate', (noun) 'the earlier part of a dramatic poem'.

[19] *The Risings and Settings* 2.6. This work (late fourth century B.C.?) is considered to be one of the earliest extant mathematical treatises.

course of the *protasis*' (which is then a completely stylistic reading) or 'the assertions derived from the *protasis*' (which is a completely conceptual reading). I do not think we should decide between these options.

Another case comes from Euclid's *Elements*. In a rare moment, the author of the *Elements* gets tired of the tedious repetition of the enunciation by the conclusion but, instead of scrapping it altogether (which he often does), he offers this compromise:[20] 'And therefore, if four lines are proportional, and what follows in the *protasis*; which it was required to prove'. This perhaps seems to refer to the stylistic unit, but again the translation is not clear. The crucial words are 'and what follows in the *protasis*', καὶ τὰ ἑξῆς τῆς προτάσεως, and the whole sentence may also be translated as: 'And therefore, if four lines are proportional, also what follows of the *protasis* <is true>; which it was required to prove', and this is clearly conceptual. But again, I think it is essentially misleading to decide between the options. We see mathematicians referring to that thing—'the start' or 'the claim' of the proposition. Both aspects are referred to by the same word. Of course the word could be taken to mean just the stylistic aspect. But was it specifically coined, at some stage, to mean that stylistic unit? This is doubtful, because it is too easily used, in contexts which are too close to the mathematical usage, to mean clearly different objects.

In the first place, while it is difficult to distinguish between the stylistic and conceptual readings, taken *per se*, they can be distinguished clearly when they appear in relation with other terms. The enunciation as stylistic unit is one of six parts, the enunciation as conceptual unit is part of a binary structure. It is the claim to be proved, the promise to be fulfilled. The structure, then, is not that of six parts, but two: promise and fulfilment. This is clearly Archimedes' use of the term. In both the introduction to the *Sphere and Cylinder*, Book 2, and the introduction to the *Method*, Archimedes has expressions such as (Introduction to *Sphere and Cylinder* 2): 'You have asked me before to write to you the proofs of those problems, whose *protaseis* I myself had sent to Conon'.

This was Archimedes' practice: to send out puzzles without solutions. He expected other mathematicians to come up with their own solutions to those puzzles. Such solutions are what he calls here 'proofs', using the term *apodeixis*. But 'proofs' here do not refer to the stylistic unit, for what Archimedes now sends has all the other parts, too (although they are scrambled together in all manner of ways). 'Proofs' here mean 'a solution to a puzzle' and *protasis* means a puzzle: it is the statement of the puzzle, without its solution.

I now move to a source which is less directly mathematical, but is much earlier in time. This is Aristotle's *Prior Analytics*, where the term *protasis* is put to massive use. There is no trace of the mathematical sense. The reference is to a conceptual unit, which can mean either 'a proposition' (in the logical, not the mathematical sense) or, more specifically and much more often, 'a premise'.[21] The Proclean meaning is that which we set out to prove; Aristotle's meaning is 'that which we use as a premise, inside a proof'. The same sense is meant when Aristotle discusses mathematical practice, in *Metaphysics* 1089a23.[22]

[20] *Elements* 11.37—i.e. towards the end of the *Elements*. There is a similar case in *Elements* 11.35.

[21] See, for example, *Prior Analytics* 24a16–17 for the general sense, 34a17–19 for the special sense.

[22] A reference related to *Metaphysics* 1089a23 is from the *Prior Analytics* 49b35. Reading the *Metaphysics* passage in light of the *Prior Analytics*, it seems that Aristotle's example of a

It is clear that Aristotle was fascinated by the mathematical format. To name the single most important example, he borrowed from mathematics the use of letters as labels. This is most apparent in the *Prior Analytics*, a work as replete with 'A, B, Γ' as any piece of mathematics. Einarson, perhaps overdoing the case, has argued for the influence of mathematics on the terminology of the *Prior Analytics*.[23] Whatever our precise judgement on the issue, it is clear that Aristotle, in the *Prior Analytics* and elsewhere, has contemporary mathematical practice near the centre of his attention.

I am not trying to say that mathematicians of Aristotle's time would not call their enunciations *protasis*; as explained above, this would have been the most natural term to use. However, it is far less likely that this was part of a wider, explicit, mathematical system of referring to parts of the proposition. If this were the case, the word *protasis* would have acquired, in the mathematical context, a specific, technical sense. And then, in a mathematical context, Aristotle would have been likely to use the word in this technical sense. But in fact, in such mathematical contexts, he uses the word with a loose, non-technical sense. His linguistic habits in general are erratic, true. He does not use the word *protasis* consistently. But what he does do regularly is to ignore the Proclean sense.[24] So it is easy to explain the mathematical usage of *protasis* on the basis of organic development, as a natural extension of common meanings. It is more difficult to imagine an explicit introduction of this usage.

But 'enunciation', if anything, is a good case for the view that Proclus follows an old tradition. This is the most obvious and the most important aspect of the style—the binary distinction into claim and proof. It could not pass unnamed, and it was named, as shown by the passages above from Autolycus and from Euclid. The remaining cases are very different.

4.2 Ekthesis *(setting-out)*

The case of *ekthesis* is clear-cut. It does not occur in any mathematical context with the sense given to it by Proclus. The cognate verb, ἐκτίθημι, *ektithemi*, does occur, with a precise and very different meaning. This last use is common in Greek mathematics, and examples can be found in the first book of the *Elements*, e.g. 1.22: 'Let some line, ΔE, *be set out*, limited at Δ but unlimited at E'.

The sense of the verb is that the geometrical object is not just any object (where you would use 'let it be'), but, instead, it must follow a definite description. Most often the geometrical object will be *given* (by magnitude, position, or both). So this is much more specific than Proclean setting-out. Indeed, the example above is taken not from the stage of setting-out, but from the stage of construction. This is to be expected, and most occurrences of this verb are not in the setting-out (where objects are usually

mathematical *protasis* would be something like 'this line is one foot long'. If anything, this is Proclean setting-out, *ekthesis*.

23 B. Einarson, 'On certain mathematical terms in Aristotle's logic', *American Journal of Philology* 57 (1936), 33–54, 151–172.

24 It is my impression—but here I am even more cautious than elsewhere—that later commentators on Aristotle do not seem to point this discrepancy between Aristotle's and the mathematicians' terminology. This is the most natural comment to be made by a late commentator. If proved, therefore, this would be a conclusive negative proof that, even in late antiquity, the Proclean scheme was not seen as 'the mathematical usage', but was merely an *ad hoc* extension of terminology developed elsewhere. (The same, of course, could be said about the further evidence I shall bring forward from Aristotle concerning other parts of the proposition.)

described in more simple ways), but in the construction. The verb means something which is different from the Proclean sense, and it covers acts that occur most often not in the Proclean setting-out, but in the Proclean construction. In typical fashion, Proclus makes no comment upon this discrepancy.

The mathematicians use *ekthesis* in a sense different from Proclus' and so, again, does our important early source, Aristotle. *Ekthesis* is a central technical term in Aristotelean logic, though what it is a technical term of is unclear, and I will not attempt to enter this controversy.[25] The one relatively clear thing is that Aristotle does *not* mean the stage of his logical arguments, where a general formulation is translated into an 'A–B–Γ' expression. To explain briefly: Aristotle proves in the *Prior Analytics* such general logical theorems as (I paraphrase 25a6): 'The universal negative necessarily converts'.

These general, letter-less theorems (the equivalent of Proclean enunciations) are proved, among other things, by lettered proofs. In this, it is sometimes possible to identify a stage which is comparable to the Proclean setting-out + definition of goal. For instance, to continue the same example (I paraphrase 25a14): 'For let AB be a universal negative premise'.

Finally, in some cases, Aristotle sometimes clarifies his arguments by reference to actual examples (e.g. in this case, 25a25, taking A to mean 'man'), and then making sense of the resulting sentences.

So we have three types of objects: (A) is the general logical subject matter; (B) is the lettered subject matter of the individual proof; and (C) is the concrete example. Mathematical, Proclean *ekthesis* has to do with the relation between the (A) general and the (B) lettered. The one relatively clear thing about Aristotelean *ekthesis* is that it has to do *not* with the relation between (A) general and (B) lettered but with the relation between (B) lettered and (C) concrete example. There is nothing extraordinary about this Aristotelean usage. Again, *ekthesis* is a highly polysemic term, literally meaning, indeed, 'setting out'. It is suitable for the setting-out of concrete examples. But the point is that, had Aristotle been aware of the Proclean sense, he would be much less inclined to use *ekthesis* the way he did, since in the Proclean sense, *ekthesis* is not something he did only rarely. In the Proclean sense, almost all Aristotelean logical arguments involve an *ekthesis*. It is only rarely that Aristotle brings in (C) concrete examples, and only occasionally is the argument related to those concrete examples, and only then we have the *ekthesis*. But the transition from (A) general case to (B) lettered presentation is ubiquitous.

Aristotelean *ekthesis* may perhaps mean both directions of the (B) lettered/(C) concrete transformation (from a (B) lettered text to a (C) concrete example, or from a (C) concrete example to a (B) lettered text). When it is the transformation from some (C) concrete example to a (B) lettered text, this is speciously similar to the Proclean *ekthesis*. It is still different (it is not the transformation from the (A) general case to the (B) lettered text), but it is in a way similar. It is therefore especially interesting to note a passage where Alexander comes close to discussing this similarity. In his commentary on *Prior Analytics* 49b32 (379.12–380.27), he reads a certain Aristotlean use of the word *ekthesis* to mean the transformation of a concrete example into letters. In this particular point he is probably wrong, but this is less interesting. More interesting are his statements that this procedure is comparable to the

[25] See M. Mignucci, 'Expository proofs in Aristotle's Syllogistic', *Oxford Studies in Ancient Philosophy* (1991), suppl., 9–28, with references to earlier literature.

geometrical use of a *diagram* (e.g. 379.28–9). That is, when looking for a mathematical analogue for the Aristotlean *ekthesis*, and specifically when discussing the use of letters, Alexander takes as his analogue not the Proclean mathematical *ekthesis*, but instead he takes the diagram. Had he been aware of the Proclean terminology, he would certainly say, instead, something such as 'and *ekthesis* is also a *mathematical* procedure'.

To sum up then: mathematicians and mathematical commentators before Proclus seem unaware of this Proclean term. They do use the cognate verb, but with a different meaning, both more specific (it is a specific type of operation) and more wide (it occurs in both Proclean setting-out and construction). Aristotle and Aristotelean commentators use the term, in a context very similar to the mathematical, in a way which is perhaps, itself, inconsistent. However, their meanings consistently differ from the Proclean sense, and betray a consistent ignorance of it. Their meanings have a family resemblance to the Proclean use—but then this is not surprising, given that they use, after all, the same Greek word, with a very similar subject-matter.

Or put it like this. Suppose you are an ancient commentator, aware of the Aristotelean literature. Suppose you look for a term for what Proclus calls the 'setting-out'. Then *ekthesis* would be a natural choice. It is not precisely the same as Aristotlean *ekthesis*, but no other Aristotlean term can fit the concept so well.

This hypothesis certainly saves the appearances, in this case. Could a similar hypothesis account for the remaining terms?

4.3 Diorismos *(definition of goal)*

Suppose you are given three lines, and you are asked to construct a triangle with them. Can this be done by Euclidean methods? The answer is, perhaps surprisingly, 'sometimes'. The triangle can be constructed only when a certain condition is fulfilled. This is because in any triangle, any two lines taken together are greater than the remaining line (Euclid proves this in *Elements* 1.20). So if the given three lines have the lengths 10, 2, and 1, the triangle cannot be constructed. The three given lines must be such that any two taken together are greater than the third; there is a limitation on the conditions of solvability. Not all problems have limitations on the conditions of solvability, but when they have them, those limitations ought to be stated explicitly. For instance, the enunciation of *Elements* 1.22 reads: 'To construct a triangle from three lines (which are equal to three given <lines>); *and it is required that, however taken together, two <lines> are greater than the remaining <line>*'.

The italicized words are what Greek mathematicians call a *diorismos*. This occurs as part of the Proclean *protasis* (and is later mirrored within the Proclean *ekthesis*), but it is completely distinct from the Proclean *diorismos*. The word *diorismos*, in this non-Proclean mathematical sense, is attested in Archimedes and Apollonius (as well as in later authors).[26] The Proclean sense is never attested prior to Proclus himself.

The general semantic field of *diorismos* is more limited than that of the preceding terms. ὁρίζω, *horizo*, is 'to draw a boundary, to limit'. The preposition *dia* serves here to stress the idea of limitation, with the implication of a more thoroughly drawn boundary, a more distinct distinction. It is a philosopher's term, used by Plato and by

[26] Archimedes, *Sphere and Cylinder* 2.4 (190.25); Apollonius, Introduction to *Conics* 4 (2.17). Among later authors are Pappus (7.5), Eutocius (Commentary on *Sphere and Cylinder* 2.4), and, indeed, Proclus himself (Commentary on 1.22).

Aristotle,[27] always in the senses of 'definition', 'distinction'. It is clear that both Proclus' and the mathematicians' use of *diorismos* can be fitted within this semantic field. Neither is an extension of the usage of the other; they are two separate coinages, but the evidence suggests that Proclus' coinage came later.

4.4 Kataskeue (construction)

Of the six Proclean terms for parts, this is the only one to appear in the Eudemean fragment conserved by Simplicius (*In Aristotelis Physica*, 61.1–68.32). It is difficult to say what here is by Simplicius (sixth century A.D.) and what is by Eudemus (late fourth century B.C.), but the following words stand a chance of being Eudemus:[28]

προγράψας τοιόνδε τι ὁ Ἱπποκράτης τοῦτο κατεσκεύασεν
Having made such a preliminary construction, Hippocrates *produced* this <case>

The funny thing is that the Greek verb *kateskeuasen* (cognate with *kataskeue*) is not translated by 'construction' but by the complementary operation 'produced'. The *kataskeue*, here, is what happens *after* the construction—what Proclus would call the *apodeixis*, 'proof'.

The Greek word *skeuos* means 'tool', 'object', almost 'thing'. *kataskeuazo*, a related verb, is 'to prepare', 'to equip'. Anything in which things are prepared or have been prepared, in any way, can be a *kataskeue*.[29] This seems to be the way in which mathematicians approach the term. It can mean the solution of a problem, as in the excerpt from Eudemus (or from Simplicius).[30] Most often the verb is used inside the formula *τῶν αὐτῶν κατασκευασθέντων*, 'the same construction being made'. This is used to introduce a new argument, where the author assumes the same diagram as before.[31] The reference of this expression, however, is definitely not to the Proclean *kataskeue*, but rather to the combination of the Proclean *ekthesis* (setting-out) and *kataskeue* (construction). The reference of the term is to the entire constructive mode, be it before or after the definition of goal. We can say therefore that, for the mathematicians, there is a distinct entity combined of the setting-out and the construction. This is where verbs are used in the imperative mode, and where the diagram is verbally reconstructed. Within this entity, the verb *ekkeistho*, cognate with *ekthesis*, is likely to be used, but it is likely to be used anywhere within this entity, both in the Proclean *ekthesis* and Proclean *kataskeue*. This entity is referred to as *kataskeue*. But then, again, it is the entity composed of both Proclean *ekthesis* and *kataskeue* which is called, by the mathematicians, *kataskeue*. The usage does not reflect stylistic units, 'before and after definition of goal', but it reflects the more substantial, conceptual unit: the constructive (as opposed to the argumentative) part of the proof.

[27] For example, Plato's *Timaeus* 38c6, Aristotle's *On Coming to Being and Passing Away* 323a22.

[28] Simplicius, *In Aristotelis Physica* 64.8–11. The best philological analysis of this difficult passage remains O. Becker, 'Zur Textgestellung des Eudemischen Berichts über die Quadratur der Möndchen durch Hippokrates von Chios', *Quellen und Studien zur Geschichte der Mathematik, Astronomie und Physik* 3 (1936), 411–19, which includes the above quotation in Eudemus' fragment.

[29] LSJ offers, for example, 'fixed assets', 'device, trick', 'system of gymnastic exercise'.

[30] In a passage to which LSJ itself refers, Pappus 174.17, *organike kataskeue* means 'solution with a mechanical tool' (one should ignore however the translation offered by *LSJ*, probably by Heath—LSJ assumes that Proclus' scheme is a true description of any mathematical practice, and tailors its mathematical examples accordingly).

[31] For example, Euclid's *Elements* 3.14, 204.11.

4.5 Apodeixis *(proof)*

The verb *deiknumi*, 'to show', came in the fifth century at the latest to have the specialized meaning 'to prove'.[32] There is no real difference between *deiknumi* and the compound verb *apodeiknumi*,[33] from which the noun *apodeixis* derives. Both words (and other related forms) are very common in many Greek writings: this is a society in which argument and persuasion occupy a central role.[34] When words are used very often, in many contexts, they are perhaps less likely to develop specific, technical senses. At any rate Greek mathematicians use both verbs (and related nouns), in expressions such as 'similarly we shall prove that . . .' (*deiknumi*: e.g. Euclid's *Elements* 1.14, 38.23), 'it has been proved that . . . (*apodeiknumi*: e.g. Archimedes' *Sphere and Cylinder* 1.24, 96.27). It is true that, in mathematical contexts, there is no evidence for *apodeiknumi* or *apodeixis* prior to Archimedes (i.e. Euclid always uses *deiknumi* or *deixis*). One could argue that *apodeixis* is a later introduction to mathematics (hence it cannot represent an early explicit codification), but I have reached the stage where I can afford to be generous, and in fact this is not a very strong argument. *apodeixis* is well attested in early sources, and Euclid's failure to use it is a quirk of his style, no more.

What is clear is that the verb and the noun always refer to the conceptual unit, not to the stylistic unit. Consider, for instance, the Euclidean phrase at the end of the Conclusion: 'Which it was required to prove'. This refers to the general conclusion, not to the particular result reached in the proof. It is about the logical content of the proposition taken as a whole, not about the part of the proposition which Proclus calls 'proof'. As I have mentioned already when discussing *protasis*, Archimedes uses *apodeixis* as the complement to *protasis*. It is what the mathematician does—as opposed to what he claims to be able to do.

There is a big difference between the enunciation and the proof. The enunciation is the most notable stylistic element of the proposition: reference to it *qua* stylistic element is natural, almost necessary. The proof, as a stylistic unit, is non-existent; it is the 'none of the above' part. By its nature, proof has much less stylistic coherence to it, but, on the other hand, it is natural, almost inevitable, to refer to the proof *qua* conceptual unit. *Qua* conceptual unit, proof is the most important thing about the proposition. It is therefore unsurprising to find that the term *apodeixis* (together with cognates) is used to refer to conceptual, not to stylistic units.

4.6 Sumperasma *(conclusion)*

This is perhaps the most 'technical' term, the one with the narrowest field of application. The related verb may have been used, from the fifth century onwards, with such diverse meanings as 'accomplish jointly', 'decide finally',[35] but the sense of 'conclusion in an argument' is very common from Aristotle onwards (e.g. *Prior Analytics*, 30a5). It is true that the noun can mean as little as 'ending' (which may perhaps fit the stylistic unit), but the sense of 'logical conclusion' is so entrenched in the scientific usage, and so relevant to mathematics, that any *sumperasma* in a mathematical context is very naturally read as 'conclusion'. Not that there are many mathematical uses. Mathematicians may speak about proofs, but they do not need to refer to conclusions that often. In a treatise which is more oriented towards

[32] LSJ quote authors such as Euripides (*Orestes* 802) and Thucydides (4.73).
[33] Attested in the fifth century, again, e.g. Aristophanes (*Clouds* 1334).
[34] See G. E. R. Lloyd, *Magic, Reason and Experience* (Cambridge,1979), pp. 102–25.
[35] LSJ entries, both from Euripides' *Medea* (887, 341, respectively).

second-order discourse, Archimedes' *Method*, Archimedes says, following the first proposition (I paraphrase lines 438.16–18): 'So this is not yet proved, yet it creates an impression that the conclusion (*sumperasma*) is true'. This refers to a proposition in which there is no *sumperasma* in the Proclean sense. The reference is to the conceptual conclusion, not to a stylistic element, simply because there is no stylistic element to refer to. Since the practice of the Proclean *sumperasma* does not occur outside Euclid, there can be no reference to it outside Euclid and his commentators. Euclid does not use the word; it remains for Proclus to apply it to Euclid's stylistic practice.

5. SUMMARY

The Proclean scheme fits Euclid's practice, roughly, but it does not fit the more general Greek mathematical practice, and its terminology cannot be found before Proclus himself. The natural interpretation of this, then, is that it was invented for the purpose of a commentary on Euclid. Perhaps, indeed, it was invented by Proclus himself.

We confront here a very general question concerning the relation between practice, on the one hand, and reflections upon practice, on the other hand. Parallels to the case discussed in this paper can be found elsewhere, and in general—as ought to have been expected—the practice comes before the reflection. Consider the ancient analysis of the structure of the tragedy: we have early evidence for the terminology for the parts of tragedy,[36] but Aeschylus' earliest plays did not conform to the pattern at all; most probably, the pattern has evolved, and was not introduced by a *fiat*. Or take the case of the modern sonata form: this was established by nineteenth-century music theorists, coming after the event. And it was established essentially as an analysis of Beethoven's canonical works—much in the way in which Euclid was taken by the author of the Proclean scheme.[37] True, with both tragedy and sonata, the reflection upon the practice came almost immediately after the practice itself. But this is to be expected: tragedy and sonata were, respectively, among the main forms of public culture in Athens and Vienna of their time. Greek mathematics, a much less popular form, had to wait longer for its theorists.

But I am rushing forward too quickly. What kind of theorists were they? Who introduced the Proclean scheme? Put it this way: what does the Proclean scheme *do*? What is its function within Proclus' treatise?

First of all, the scheme serves as the springboard for an extensive discussion of the philosophy of mathematics, in the commentary to the first proposition of the *Elements*. Proclus has much to say on the very nature of mathematics and geometry, in the two prologues respectively. Later on, in the course of his commentaries on individual definitions or propositions, Proclus often provides comments of a philosophical nature.[38] But his only extensive discussion of philosophical issues which arise from the practice of mathematicians is in 200.22–213.11, preceding his mathematical commentary on proposition 1. The bulk of this is the Proclean scheme and its philosophical commentary (203.1–210.25). Having given the scheme in outline, Proclus discusses questions such as: 'What is "to be given"?' (205.13–206.11),[39] 'What

[36] Aristophanes, *Frogs* 1119, refers to the first part, the *prologos*. Aristotle, *Nicomachean Ethics* 1123a21, refers to the second, *parodos*. Both seem to refer to the stylistic unit.

[37] C. Rosen, *Sonata Forms* (New York, 1988).

[38] For example, in the beginning of his commentary to 1.20 (322.4–323.3), Proclus discusses the philosophically interesting question: 'why prove an obvious result?'.

[39] This question arises from Proclus' definition of the *protasis* and *ekthesis*.

is the nature of a mathematical proof?' (206.11–207.3). In short, for Proclus himself, the scheme functions as a way of identifying the philosophical issues arising from mathematics. Here is a commentator, breaking down his subject-matter to its components, and then making comments on each individual component.

Proclus' commentary has an institutional context, and is not just Proclus' private musings. Among other things, the treatise was used for teaching in the Neo-Platonist Academy.[40] What was the function of Proclus' scheme in this institutional context? Proclus could not have been more explicit about that. Following the general discussion of the parts of the proposition, Proclus shows how the parts are found in the first proposition, and finally sums up: 'The student should do this also for the remaining propositions . . . for a comprehensive survey of these matters will provide no little exercise and practice in geometrical reasoning'.[41]

This is the only piece of homework Proclus sets down in the commentary. He does not return to it explicitly, but here and there he gives, as it were, gentle hints.[42] And this is consistent with the use Proclus himself makes of the scheme. For a philosophically motivated student, looking for the parts of the proposition is not too demanding in terms of mathematical expertise. And it is of some philosophical interest. At least, it makes the reader follow the larger structure of reasoning—to look for proofs, as distinct from enunciations and constructions.

This is in line with the nature of the terminology. The terminology is, in the case of *diorismos*, incompatible with the established mathematical practice. The distinction between the terms *ekthesis* and *kataskeue*, coming from mathematics, is unnatural. But looked at from the perspective of philosophy, especially in the Platonist–Aristotelean tradition, the terminology becomes obvious. *Ekthesis* makes most sense as an extension of a key Aristotelean logical term to mathematics. *Diorismos* and *sumperasma* are not confined to Aristotle, but they are common Aristotelean logical terms. Such arguments cannot be conclusive, but the emerging picture is coherent: we see a reader of Aristotle, a teacher of a class in Mathematics for Philosophers.

Do we see Proclus, then? Is he the inventor of the scheme? I do not know, and he does not say so. The identity of the individual is less important then the type of context in which the scheme could arise, and I believe it is very probable that the scheme was introduced by commentators on Euclid, whose interests were mostly philosophical. That this happened in a relatively late period is therefore almost certain.

The system is ingenious. It does not fit the practice precisely—but no *post factum* scheme could. It is thus an example of the sensitive, original reading of ancient books by late commentators. As for the mathematicians themselves, it seems that they did not have this scheme in front of them while composing their works. Some other mechanism, explaining the relatively fixed style of Greek mathematics, must be found. And this mechanism must take account of this: that no style manual was ever produced for early Greek mathematics.

Gonville and Caius College, Cambridge

REVIEL NETZ
rnetz@dibinst.mit.edu

[40] See I. Mueler, foreword to second edition of *Proclus: A Commentary on the First Book of Euclid's Elements*, trans. G. Morrow (Princeton, 1992), pp. xlviii-l.

[41] I follow Morrow's translation, from 210.18–25.

[42] 299.5–11 (paraphrasing): 'not all the parts are present here, e.g. we do not have the construction—but the proof, which is indispensable, is present'. Are the students expected to check the other four parts?